

Interaction of a Magnet and a Point Charge: Unrecognized Internal Electromagnetic Momentum Eliminates the Myth of Hidden Mechanical Momentum

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Abstract

A model calculation using the Darwin Lagrangian is carried out for a magnet consisting of two current-carrying charges constrained by centripetal forces to move in a circular path in the presence of the electric field from a distant external point charge. In the limit that the magnet's two charges are *non-interacting*, the calculation recovers the only valid calculation for “hidden *mechanical* momentum.” However, if the magnet's charges are mutually *interacting*, then there is internal *electromagnetic* linear momentum associated with the perturbed magnet's electrostatic charge distribution and the motion of the magnet's charges. This internal *electromagnetic* momentum does not seem to be recognized as distinct from the familiar external electromagnetic momentum associated with the electric field of the external charge and the magnetic field of the unperturbed magnet. In the multiparticle limit, the “hidden *mechanical* momentum” becomes negligible while the internal *electromagnetic* momentum provides the compensating linear momentum required by the relativistic conservation law connecting the total linear momentum to motion of the center of energy. Whereas the changes in the external electromagnetic momentum are often associated with *magnetic* forces of order $1/c^2$, changes in the internal electromagnetic momentum are associated with the *electrical* forces of order $1/c^2$. These electrical forces are relevant to the Shockley-James paradox and to the experimentally observed Aharonov-Bohm and Aharonov-Casher phase shifts.

I. INTRODUCTION

A. Problem of the Interaction of a Magnet and a Point Charge

Ever since Aharonov and Bohm's claim[1] that a charged particle could be influenced by magnetic fluxes in the absence of electromagnetic forces, there has been increased interest in the interaction of a magnet and a point charge. In connection with this interaction, Shockley and James[2] introduced the idea of "hidden momentum" and Coleman and van Vleck[3] suggested this "hidden momentum" was entirely mechanical in nature. The idea of "hidden mechanical momentum" now appears in leading textbooks of electromagnetism.[4][5]

However, despite an extensive literature invoking "hidden mechanical momentum,"[6] there still remains only one valid calculation for this quantity in the literature. This calculation involves a current loop modeled as a single charged particle (or equivalently many *non-interacting* particles) moving in a fixed orbit under the influence of an external electric field generated by the external point charge. This one-particle calculation was given by Penfield and Haus,[7] is sketched in a foot note in the work of Coleman and van Vleck,[3] and today appears as an example[8] in an excellent undergraduate electromagnetism textbook. All the remaining references to hidden momentum represent empty claims unsupported by any calculation.

In the present article, we move beyond the one-moving-particle magnet over to a two-*interacting*-moving-particle magnet. Just as the one-particle magnet allows a complete calculation of the system linear momentum, so too this two-interacting-moving-particle magnet allows complete calculation of the system linear momentum using the Darwin Lagrangian valid to order v^2/c^2 . The *interacting* multiparticle model for the magnet introduces an aspect which does not seem to have been recognized previously in the literature. The mutual interactions of the charges of the magnet in the perturbing presence of the external electric field produce an internal *electromagnetic* linear momentum. In the limit of a large number of charges (where the electromagnetic interactions dominate the mechanical aspects of the magnet's behavior), the hidden *mechanical* momentum of the noninteracting-particle magnet vanishes and is replaced by an equal amount of internal *electromagnetic* momentum. Thus the physicists who insisted upon a vanishing total linear momentum for a relativistic system with a stationary center of energy (and vanishing power from external forces) are

indeed noting a demand of relativistic theory; however, the identification of the unrecognized linear momentum as "*mechanical*" seems entirely misdirected. Just as the *mechanical* kinetic energy in a current-carrying solenoid can essentially always be ignored compared to the *electromagnetic* energy, so the *mechanical* linear momentum of a multiparticle magnetic can essentially always be ignored compared to the internal *electromagnetic* linear momentum in the presence of an external electric field.

In an attempt to correct the misdirected hidden-mechanical-momentum claims in the electromagnetism literature, the present article presents the elementary calculation for the internal electromagnetic momentum of a two-interacting-moving-particle magnet in the presence of an external electric field due to an external point charge. The change from hidden *mechanical* momentum over to internal *electromagnetic* momentum has crucial implications for the forces on a charged particle passing a solenoid and hence upon the proper interpretation of the experimental observations for this interaction. However, the discussion of the unrecognized classical electromagnetic forces responsible for the Aharonov-Bohm[1] and Aharonov-Casher[9] phase shifts will be reserved for another publication.

B. Outline of the Presentation

We start our analysis by reviewing the Darwin Lagrangian and the expression for the canonical momentum associated with each charged particle. Then we note the relativistic conservation law connecting the system linear momentum and the velocity of the system center of energy. Having reviewed these preliminaries, we consider a model for a magnet which has charged particles constrained by centripetal forces to move in a circular path (corresponding to charged beads sliding on a frictionless ring) while a neutralizing particle is held at the center of the circular path. A distant external particle is held at a fixed location by external forces of constraint. Next we calculated the electromagnetic field momentum involving the electric field of the external charge and the magnetic field of the unperturbed moving charges of the magnet. After this calculation for the unperturbed situation, we consider the perturbed motion of the magnet charges due to the presence of the external charge. First we review the case of a one-moving-particle magnet (or equivalently many *non-interacting* magnet charges). We repeat the earlier calculation in the literature showing that the mechanical momentum of the magnet particle is equal in magnitude and opposite

in direction compared to the electromagnetic field momentum between the charge and the unperturbed magnet. Thus we see that the relativistic conservation law regarding the center of energy indeed holds. Then we turn to the model for a magnet which does not seem to have been treated earlier. We consider two *interacting* charges and calculate the steady-state perturbed motion in the presence of the external charge. We find that the internal linear momentum of the magnet in the presence of the external charge now has two contributions, a mechanical part and an electromagnetic part; the electromagnetic part is due to the motion of each of the charged magnet particles in the field of the other magnet particle. The relativistic conservation law is again observed. However, it is clear that as the size of the electrical charges increases, the electromagnetic momentum contribution increases and the mechanical contribution decreases. Indeed in the case of a large number of interacting magnet particles, the internal electromagnetic linear momentum in the presence of the external charge will overwhelm the mechanical momentum. In a multiparticle magnet, the “hidden mechanical momentum” is completely negligible. Finally we discuss the idea of internal electromagnetic momentum in connection with the claims in the textbook and research literature. We point out that internal electromagnetic momentum will lead to forces on the external charge which are completely different from those associated with “hidden mechanical momentum.” The recognition of these classical electromagnetic forces is important for understanding the experimentally observed Aharonov-Bohm and Aharonov-Casher phase shifts and also the role of resistivity in interactions between current loops and passing charges.

II. BASIS FOR THE CALCULATIONS

A. The Darwin Lagrangian

Although the experimentally-measured Aharonov-Bohm phase shift involving the interaction of a magnet and a point charge is usually described within *non-relativistic* quantum theory,[10] the effect is actually of order $1/c^2$. In SI units the phase shift is $q\Phi/\hbar$, whereas in gaussian units the phase shift is $q\Phi/(\hbar c)$ with the magnetic flux Φ contributing a second factor of $1/c$. Thus we expect that the interaction of a magnet and a point charge needs to be treated relativistically, at least to order v^2/c^2 . Within classical electrodynamics, the

electromagnetic interaction of point charges e_i at locations \mathbf{r}_i moving with velocity \mathbf{v}_i can be described through order v^2/c^2 by the Darwin Lagrangian[11]

$$\begin{aligned}\mathcal{L} = & \sum_{i=1}^{i=N} m_i c^2 \left(-1 + \frac{\mathbf{v}_i^2}{2c^2} + \frac{(\mathbf{v}_i^2)^2}{8c^4} \right) - \frac{1}{2} \sum_{i=1}^{i=N} \sum_{j \neq i} \frac{e_i e_j}{|\mathbf{r}_i - \mathbf{r}_j|} \\ & + \frac{1}{2} \sum_{i=1}^{i=N} \sum_{j \neq i} \frac{e_i e_j}{2c^2} \left[\frac{\mathbf{v}_i \cdot \mathbf{v}_j}{|\mathbf{r}_i - \mathbf{r}_j|} + \frac{\mathbf{v}_i \cdot (\mathbf{r}_i - \mathbf{r}_j) \mathbf{v}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right]\end{aligned}\quad (1)$$

B. Electric and Magnetic Fields from the Darwin Lagrangian

The Lagrangian equations of motion following from Eq. (1) can be rewritten in the form of Newton's second law for the mechanical momentum \mathbf{p}^{mech} and force \mathbf{F} , $d\mathbf{p}^{mech}/dt = d(m\gamma\mathbf{v})/dt = \mathbf{F}$, with $\gamma = (1 - v^2/c^2)^{-1/2}$. In this Newtonian form, we have

$$\begin{aligned}\frac{d}{dt} [m_i \gamma_i \mathbf{v}_i] &= \frac{d}{dt} \left[\frac{m_i \mathbf{v}_i}{(1 - \mathbf{v}_i^2/c^2)^{1/2}} \right] \approx \frac{d}{dt} \left[m_i \left(1 + \frac{\mathbf{v}_i^2}{2c^2} \right) \mathbf{v}_i \right] \\ &= e_i \sum_{j \neq i} \mathbf{E}_j(\mathbf{r}_i, t) + e_i \frac{\mathbf{v}_i}{c} \times \sum_{j \neq i} \mathbf{B}_j(\mathbf{r}_i, t)\end{aligned}\quad (2)$$

with the Lorentz force on the i th particle arising from the electromagnetic fields of the other particles. The electromagnetic fields due to the j th particle are given through order v^2/c^2 by[12]

$$\begin{aligned}\mathbf{E}_j(\mathbf{r}, \mathbf{t}) &= e_j \frac{(\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r} - \mathbf{r}_j|^3} \left[1 + \frac{\mathbf{v}_j^2}{2c^2} - \frac{3}{2} \left(\frac{\mathbf{v}_j \cdot (\mathbf{r} - \mathbf{r}_j)}{c|\mathbf{r} - \mathbf{r}_j|} \right)^2 \right] \\ &\quad - \frac{e_j}{2c^2} \left(\frac{\mathbf{a}_j}{|\mathbf{r} - \mathbf{r}_j|} + \frac{\mathbf{a}_j \cdot (\mathbf{r} - \mathbf{r}_j)(\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r} - \mathbf{r}_j|^3} \right)\end{aligned}\quad (3)$$

and

$$\mathbf{B}_j(\mathbf{r}, t) = e_j \frac{\mathbf{v}_j}{c} \times \frac{(\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r} - \mathbf{r}_j|^3}\quad (4)$$

where in Eq. (3) the quantity \mathbf{a}_j refers to the acceleration of the j th particle.

C. Canonical Linear Momentum from the Darwin Lagrangian

The canonical linear momentum $\mathbf{p}_i = \mathbf{p}_i^{mech} + \mathbf{p}_i^{field}$ associated with the i th charge is given by

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = \mathbf{p}_i = m_i \left(1 + \frac{\mathbf{v}_i^2}{2c^2} \right) \mathbf{v}_i + \sum_{j \neq i} \frac{e_i e_j}{2c^2} \left(\frac{\mathbf{v}_j}{|\mathbf{r}_i - \mathbf{r}_j|} + \frac{\mathbf{v}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right)\quad (5)$$

corresponding to the *mechanical* linear momentum of the i th particle

$$\mathbf{p}_i^{mech} = m_i \gamma_i \mathbf{v}_i \approx m_i [1 + v_i^2/(2c^2)] \mathbf{v}_i = m_i \mathbf{v}_i + m_i v_i^2 \mathbf{v}_i / (2c^2) \quad (6)$$

and the *electromagnetic* linear momenta associated with the *electric* field of the i th particle and *magnetic* fields of the other j th particles

$$\mathbf{p}_i^{field} = \sum_{j \neq i} 1/(4\pi c) \int d^3r \mathbf{E}_i \times \mathbf{B}_j = \sum_{j \neq i} \frac{e_i e_j}{2c^2} \left(\frac{\mathbf{v}_j}{|\mathbf{r}_i - \mathbf{r}_j|} + \frac{\mathbf{v}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right) \quad (7)$$

Thus the total linear momentum \mathbf{P} is given by

$$\mathbf{P} = \sum_i \mathbf{p}_i = \sum_i m_i \left(1 + \frac{v_i^2}{2c^2} \right) \mathbf{v}_i + \frac{1}{2} \sum_i \sum_{j \neq i} \frac{e_i e_j}{2c^2} \left(\frac{\mathbf{v}_j}{|\mathbf{r}_i - \mathbf{r}_j|} + \frac{\mathbf{v}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right) \quad (8)$$

D. Relativistic Conservation Laws

The Darwin Lagrangian satisfies the familiar conservation laws involving energy, linear momentum, and angular momentum. Through order v^2/c^2 the Darwin Lagrangian satisfies the last (and only specifically relativistic) conservation law for relativistic systems involving the uniform motion of the center of energy. This last relativistic conservation law takes the general form[13]

$$\sum_i (\mathbf{F}_{ext\ i} \cdot \mathbf{v}_i) \mathbf{r}_i = \frac{d}{dt} (U \vec{\mathcal{X}}_E) - c^2 \mathbf{P} \quad (9)$$

connecting the external forces $\mathbf{F}_{ext\ i}$ on the particles to the time-rate-of-change of the system energy U times the center of energy $\vec{\mathcal{X}}_E$ and the total system momentum \mathbf{P} . We see that when there is no power introduced by external forces on the system (so that the left-hand side of Eq. (9) vanishes and the system energy U is conserved), then the relativistic conservation law reads

$$U d\vec{\mathcal{X}}_E/dt = c^2 \mathbf{P} \quad (10)$$

In this case, if the center of energy is at rest, $d\vec{\mathcal{X}}_E/dt = 0$, then the total system momentum \mathbf{P} must vanish. It is this conservation law to which physicists appeal when introducing the idea of hidden mechanical momentum. Indeed, Jackson has a problem[14] in his graduate text which considers a point charge at the center of a toroidal magnet. There are no external forces on the system, the center of energy is not moving, and therefore the total linear momentum of the system must vanish. The text then suggests that a hidden mechanical momentum is present in the magnet. Also, in his undergraduate text, Griffiths has an

example[8] of a rectangular current loop in an external electric field, and again suggests that hidden mechanical momentum is present in the current loop. Although both these examples involve correct calculations, the magnets are treated as though they were one-moving-particle magnets (or *non-interacting*-moving-particle magnets) since there is no interaction allowed between the particles of the magnets. This crucial restriction to non-interacting particles is contrary to the whole spirit of multiparticle electromagnetism where the mutual interactions between charges lead to such behaviors as electrostatic screening and self-induction.

III. BASIC MODEL FOR THE MAGNET-POINT CHARGE INTERACTION

A. Magnet Modeled by Moving Point Charges

The model for a magnet used in our calculations involves N charges e_i of charge e which are held by external centripetal forces of constraint in a circular orbit of radius R centered on the origin in the xy -plane while an opposite neutralizing charge $Q = -Ne$ is located at the origin. The charges e_i in the circular orbit are free to move along the orbit due to any electromagnetic forces which are tangential to the orbit. Thus in essence, the magnetic is pictured as charged beads e sliding on a frictionless ring in the xy -plane with a balancing opposite charge Q located at the center of the ring.

In the absence of any perturbing influence, the charges e_i can be equally spaced with initial phases ϕ_i around the circular orbit and move with angular velocity ω_0 , speed $v_0 = \omega_0 R$, displacement

$$\mathbf{r}_{0i} = R[\hat{x} \cos(\omega_0 t + \phi_i) + \hat{y} \sin(\omega_0 t + \phi_i)] = \hat{r}_{0i} R \quad (11)$$

and velocity

$$\mathbf{v}_{0i} = \omega_0 R[-\hat{x} \sin(\omega_0 t + \phi_i) + \hat{y} \cos(\omega_0 t + \phi_i)] = \hat{\phi}_{0i} \omega_0 R \quad (12)$$

Here the radial and tangential unit vectors for the charge e_i are

$$\hat{r}_{0i} = \hat{x} \cos(\omega_0 t + \phi_i) + \hat{y} \sin(\omega_0 t + \phi_i) \quad (13)$$

and

$$\hat{\phi}_{0i} = -\hat{x} \sin(\omega_0 t + \phi_i) + \hat{y} \cos(\omega_0 t + \phi_i) \quad (14)$$

The magnetic moment $\vec{\mu}$ of the N -moving-particle magnet is given by the time-average of

$e\mathbf{r} \times \mathbf{v}/(2c)$ corresponding to

$$\vec{\mu} = \left\langle \sum_{i=1}^N e_i \mathbf{r}_i \times \mathbf{v}_i / (2c) \right\rangle = \hat{z} N e R^2 \omega_0 / (2c) \quad (15)$$

We now introduce an external charge q located on the x -axis at coordinate x_q , $\mathbf{r}_q = \hat{x}x_q$, which is held in place by external forces of constraint. If the charge q is far away from the magnet, $R \ll x_q$, then the electric field $\mathbf{E}_q(\mathbf{r})$ near the position of the magnet, $r \approx R \ll x_q$, is given by

$$\mathbf{E}_q(\mathbf{r}) = \frac{q(\mathbf{r} - \mathbf{r}_q)}{|\mathbf{r} - \mathbf{r}_q|^3} \approx \frac{q(\mathbf{r} - \hat{x}x_q)}{x_q^3} \left(1 + \frac{3\hat{x} \cdot \mathbf{r}}{x_q} \right) = \frac{-\hat{x}q}{x_q^2} + \frac{q[\mathbf{r} - 3\hat{x}(\hat{x} \cdot \mathbf{r})]}{x_q^3} \quad (16)$$

to order r/x_q . In this article, we will need only the leading term in the electric field, $\mathbf{E}_q(0) \approx -\hat{x}q/x_q^2$.

B. Familiar Electromagnetic Field Momentum for an Unperturbed Current Loop in an External Electric Field

In the approximation that the charges e_i of the magnet are not perturbed by the presence of the point charge q , we can obtain the linear momentum in the electromagnetic field as the contribution of the electric field of the point charge q and the magnetic field arising from the moving magnet charges e_i . This correspond to the canonical momentum of the stationary charge q (which is entirely electromagnetic field momentum),

$$\begin{aligned} \mathbf{p}_q &= \mathbf{p}_q^{field} = \sum_{i=1}^N \frac{1}{4\pi c} \int d^3r \mathbf{E}_q \times \mathbf{B}_i = \sum_{j=1}^N \frac{qe_j}{2c^2} \left(\frac{\mathbf{v}_j}{|\mathbf{r}_q - \mathbf{r}_j|} + \frac{\mathbf{v}_j \cdot (\mathbf{r}_q - \mathbf{r}_j)(\mathbf{r}_q - \mathbf{r}_j)}{|\mathbf{r}_q - \mathbf{r}_j|^3} \right) \\ &\approx \sum_{j=1}^N \frac{qe_j}{2c^2} \left[\frac{\mathbf{v}_j}{x_q} \left(1 + \frac{\hat{x} \cdot \mathbf{r}_j}{x_q} \right) + \frac{\mathbf{v}_j \cdot (\hat{x}x_q - \mathbf{r}_j)(\hat{x}x_q - \mathbf{r}_j)}{x_q^3} \left(1 + \frac{3\hat{x} \cdot \mathbf{r}_j}{x_q} \right) \right] \end{aligned} \quad (17)$$

We now insert the expressions (11) and (12) into Eq. (17), average over time, and keep terms through first order $1/x_q^2$. We note that $\mathbf{v}_j \cdot \mathbf{r}_j = 0$, $\langle \mathbf{v}_j \rangle = 0$, $\langle (\mathbf{v}_j \cdot \hat{x})(\hat{x} \cdot \mathbf{r}_j) \rangle = 0$, and $\langle \mathbf{v}_j(\hat{x} \cdot \mathbf{r}_j) \rangle = \hat{y}\omega_0 R^2/2 = -\langle (\mathbf{v}_j \cdot \hat{x})\mathbf{r}_j \rangle$. The time-averaged electromagnetic field momentum \mathbf{p}_q receives equal contributions from each charge giving

$$\langle \mathbf{p}_q \rangle = \frac{\hat{y}Nqe\omega_0 R^2}{2c^2 x_q^2} = \frac{1}{c} \mathbf{E}_q(0) \times \vec{\mu} \quad (18)$$

This is the familiar expression for the electromagnetic field momentum for an unperturbed localized steady current located in an external electric field; it corresponds to a problem in Jackson's text.[15] Even if the magnet is perturbed by the electric field $\mathbf{E}_q(0)$, the field momentum in Eq. (18) will remain unchanged to lowest order in this perturbing field.

IV. “HIDDEN MECHANICAL MOMENTUM” IN THE ONE-MOVING-PARTICLE MAGNET

A. Perturbation of the One-Moving-Particle Magnet

In order to connect our new work on internal electromagnetic momentum with the previous literature involving “hidden mechanical momentum,” we start with a magnet consisting of one moving charged particle of mass m and charge e ; for this one-moving-particle magnet the index “ i ” corresponds only to $i = 1$. The neutralizing charge $Q = -e$ is located at the origin. (This one-moving-particle magnet is actually equivalent to assuming that there are N *non-interacting* moving particles.)

The position of the charged particle e in the magnet is perturbed by the presence of the external electric field due to the charge q . The phase angle is no longer the $\omega_0 t + \phi_i$ appearing in Eq. (11), but rather becomes $\omega_0 t + \phi_i + \eta_i(t)$, so that the displacement of the particle through first order in the perturbation is now

$$\mathbf{r}_i = R\{\hat{x}\cos[\omega_0 t + \phi_i + \eta_i(t)] + \hat{y}\sin[\omega_0 t + \phi_i + \eta_i(t)]\} = R\hat{r}_i = \mathbf{r}_{0i} + \delta\mathbf{r}_i \quad (19)$$

where

$$\hat{r}_i = \hat{x}\cos[\omega_0 t + \phi_i + \eta_i(t)] + \hat{y}\sin[\omega_0 t + \phi_i + \eta_i(t)] \quad (20)$$

the unperturbed displacement \mathbf{r}_{0i} is given in Eq. (11), and

$$\delta\mathbf{r}_i = \eta_i R\{-\hat{x}\sin[\omega_0 t + \phi_i] + \hat{y}\cos[\omega_0 t + \phi_i]\} = \hat{\phi}_{0i} R\eta_i \quad (21)$$

Due to the perturbation, the velocity $\mathbf{v}_i = d\mathbf{r}_i/dt$ is now

$$\begin{aligned} \mathbf{v}_i &= (\omega_0 + d\eta_i/dt)R\{-\hat{x}\sin[\omega_0 t + \phi_i + \eta_i(t)] + \hat{y}\cos[\omega_0 t + \phi_i + \eta_i(t)]\} \\ &= \hat{\phi}_i(\omega_0 + d\eta_i/dt)R = \mathbf{v}_{0i} + \delta\mathbf{v}_i \end{aligned} \quad (22)$$

where

$$\hat{\phi}_i = -\hat{x}\sin[\omega_0 t + \phi_i + \eta_i(t)] + \hat{y}\cos[\omega_0 t + \phi_i + \eta_i(t)] \quad (23)$$

the unperturbed velocity \mathbf{v}_{0i} is given in Eq. (12), and

$$\delta\mathbf{v}_i = \hat{\phi}_{0i} R d\eta_i/dt - \hat{r}_{0i} R \omega_0 \eta_i \quad (24)$$

through first order in the perturbation η_i . In obtaining Eqs. (21) and (24), we have used Eqs. (13) and (14) as well as the small-angle approximations $\cos(\phi + \delta\phi) \approx \cos\phi - \delta\phi \sin\phi$ and $\sin(\phi + \delta\phi) \approx \sin\phi + \delta\phi \cos\phi$.

The mechanical momentum in Eq. (6) involves two terms $\mathbf{p}_i^{mech} = m_i \mathbf{v}_i + m_i v_i^2 \mathbf{v}_i / (2c^2)$. When averaged in time, the first term vanishes, $\langle m_i \mathbf{v}_i \rangle = m_i \langle \mathbf{v}_i \rangle = 0$, since for a stationary situation the time-average velocity vanishes. Consequently, the average mechanical momentum in a stationary situation involves terms which already contain a factor of $1/c^2$, $\langle \mathbf{p}_i^{mech} \rangle = \langle m_i v_i^2 \mathbf{v}_i / (2c^2) \rangle$, so that the velocity \mathbf{v}_i needs to be calculated only through nonrelativistic order.

B. Calculation of the Perturbation by Energy Conservation

For the one-moving-particle magnet (or N -non-interacting-moving-particle magnet), it is convenient to obtain the perturbed phase $\eta_i(t)$ from energy conservation. The centripetal forces of constraint do no work, and hence the total energy (kinetic plus electrostatic particle energy) of the particle (or of each particle of a non-interacting group) is conserved,

$$\frac{1}{2} m v_{0i}^2 = \frac{1}{2} m v_i^2 + (eq/x_q^2) R \cos[\omega_0 t + \phi_i + \eta_i(t)] \quad (25)$$

where the last term is the potential energy of the magnet charge e in the electrostatic field of the external charge q . Now we are interested in the behavior of the system through first order in the perturbation eq/x_q^2 . Thus we expand each of the terms on the right-hand side of Eq. (25). Expanding the particle kinetic energy $mv^2/2$, we have from Eqs. (12), (22) and (24)

$$\frac{1}{2} m v_i^2 = \frac{1}{2} m (\mathbf{v}_{0i} + \delta \mathbf{v})^2 = \frac{1}{2} m v_{0i}^2 + m \mathbf{v}_{0i} \cdot \delta \mathbf{v}_i = \frac{1}{2} m v_{0i}^2 + m \omega_0 R^2 \frac{d\eta_i}{dt} \quad (26)$$

Also, the term involving the cosine in Eq. (25) is already first order in the perturbation, and therefore we may drop the η_i in the argument of the cosine. The energy conservation equation then becomes

$$\frac{1}{2} m v_{0i}^2 = \frac{1}{2} m v_{0i}^2 + m \omega_0 R^2 \frac{d\eta_i}{dt} + \frac{eq}{x_q^2} R \cos(\omega_0 t + \phi_i) \quad (27)$$

or

$$\frac{d\eta_i}{dt} = -\frac{eq}{x_q^2 m \omega_0 R} \cos(\omega_0 t + \phi_i) \quad (28)$$

Integrating once, we have the perturbing phase as

$$\eta_i(t) = -\frac{eq}{x_q^2 m \omega_0^2 R} \sin(\omega_0 t + \phi_i) \quad (29)$$

C. Mechanical Momentum of the Perturbed One-Moving-Particle Magnet

Since the external charge q is not moving, the canonical momentum \mathbf{p}_e given in Eq. (5) for the one-moving-particle magnet consists entirely of mechanical momentum. We can now calculate the average mechanical linear momentum of the charge e which is moving in a circular orbit. Through first order in the perturbing force eq/x_q^2 and second order in v_0/c , the mechanical momentum \mathbf{p}_m^{mech} is given from Eqs. (6) and (22) by

$$\begin{aligned} \mathbf{p}_e &= \mathbf{p}_m^{mech} = m\gamma_e \mathbf{v}_e = (m + m\mathbf{v}_0 \cdot \delta\mathbf{v}_i/c^2)(\mathbf{v}_{0i} + \delta\mathbf{v}_i) \\ &= m(\mathbf{v}_{0i} + \delta\mathbf{v}_i) + m(\mathbf{v}_0 \cdot \delta\mathbf{v}_i)\mathbf{v}_{0i}/c^2 \end{aligned} \quad (30)$$

Then averaging in time and noting that $\langle \mathbf{v}_{0i} \rangle = \langle \delta\mathbf{v}_i \rangle = 0$, we have from Eq. (12), (29) and (30),

$$\begin{aligned} \mathbf{p}_e &= \left\langle \frac{m\gamma_0^3(\mathbf{v}_0 \cdot \delta\mathbf{v}_i)\mathbf{v}_{0i}}{c^2} \right\rangle = \left\langle \frac{m\omega_0 R^2}{c^2} \frac{d\eta_i}{dt} \mathbf{v}_{0i} \right\rangle \\ &= \left\langle \frac{m\omega_0 R^2}{c^2} \left(-\frac{eq}{x_q^2 m \omega_0 R} \cos(\omega_0 t + \phi_i) \right) \omega_0 R [-\hat{x} \sin(\omega_0 t + \phi_i) + \hat{y} \cos(\omega_0 t + \phi_i)] \right\rangle \\ &= -\hat{y} \frac{eq\omega_0 R^2}{2c^2 x_q^2} \end{aligned} \quad (31)$$

This result is just the negative of Eq. (18) when $N = 1$. Thus the average mechanical momentum (for the single charge e of the one-moving-particle magnet in the presence of the charge q) is equal in magnitude and opposite in sign from the canonical momentum \mathbf{p}_q corresponding to the familiar electromagnetic field momentum associated with the electric field of the external charge q and the magnetic field of the magnet. Thus the total momentum of the system consisting of the charge q and the one-moving-particle magnet indeed vanishes,

$$\langle \mathbf{P} \rangle = \langle \mathbf{p}_q \rangle + \langle \mathbf{p}_e \rangle = \langle \mathbf{p}_q^{field} \rangle + \langle \mathbf{p}_e^{mech} \rangle = \hat{y} \frac{eq\omega_0 R^2}{2x_q^2 c^2} - \hat{y} \frac{eq\omega_0 R^2}{2x_q^2 c^2} = 0 \quad (32)$$

as required by the relativistic conservation law (10). The mechanical linear momentum \mathbf{p}_e given in Eq. (31) is what is identified in the literature as the “hidden mechanical momentum.”

V. INTERNAL ELECTROMAGNETIC MOMENTUM IN THE TWO-INTERACTING-MOVING-PARTICLE MAGNET

A. Improved Model of Two Moving Magnet Charges which Interact

The physical magnets found in nature do not consist of a single charged particle (or of non-interacting particles) sliding on a frictionless ring. Hence we turn to a model consisting of two *interacting* moving charged particles as an improvement over our one-moving-particle model for a magnet. Now we have two charges e held by external centripetal forces of constraint in a circular orbit of radius R centered on the origin in the xy -plane while a neutralizing charge $Q = -2e$ is located at the origin. The calculation for the magnetic moment $\vec{\mu}$ and the field momentum associated with the canonical momentum \mathbf{p}_q follow as in the calculations above with the results in Eqs. (15) and (18) corresponding to $N = 2$.

B. Internal Electromagnetic Field Momentum

The total linear momentum \mathbf{P} of our example involves not only the electromagnetic field momentum $\mathbf{p}_q = \mathbf{p}_q^{field}$ associated with the canonical momentum of the stationary particle q but also the canonical momenta \mathbf{p}_{e1} and \mathbf{p}_{e2} associated with the particles of the magnet. Since the charge q is at rest, it has no magnetic field, and hence it does not contribute to the canonical momentum of \mathbf{p}_{e1} or \mathbf{p}_{e2} . However, the canonical momentum \mathbf{p}_{e1} of the first magnet particle includes both its mechanical momentum and also the electromagnetic momentum associated with its own electric field and the magnetic field of the other moving particle in the magnet. Now the *unperturbed* motion of the magnet charges given in Eqs. (11) and (12) involves no average linear momentum because the two charges are always moving with opposite velocities on opposite sides of the circular orbit. However, the *perturbed* motion will indeed involve net linear momentum for the magnet particles. As soon as the particles of the magnet have mutual interactions, then the mechanical kinetic energy changes (which provided the basis for the internal *mechanical* momentum in the one-moving-particle magnet) are suppressed as energy goes into electrostatic energy of the interacting particles. The internal *mechanical* momentum of the magnet decreases because of the electrostatic interactions, and internal *electromagnetic* momentum appears in the internal electromagnetic fields. We will illustrate this situation explicitly for our example

involving the two-interacting-moving-particle magnet.

C. Calculation of the Perturbation Using Nonrelativistic Forces

In order to obtain the internal linear momentum $\mathbf{p}_{e1} + \mathbf{p}_{e2}$ of the magnet in the presence of the external electric field \mathbf{E}_q due to the charge q , we need to calculate the perturbed motion of the particles e_1 and e_2 . The perturbed positions of the two charges e_1 and e_2 of the magnet will be written as in Eq. (19), where the unperturbed initial phases differ by π , $\phi_1 - \phi_2 = \pi$, and where it is again assumed that $\eta_i(t)$ is a small correction. The perturbed velocities of the charges are as given in Eq. (22), and the accelerations then follow as

$$\begin{aligned} \mathbf{a}_i &= (\omega_0 + d\eta_i/dt)^2 R \{ -\hat{x} \cos[\omega_0 t + \phi_i + \eta_i(t)] - \hat{y} \sin[\omega_0 t + \phi_i + \eta_i(t)] \} \\ &\quad + (d^2\eta_i/dt^2) R \{ -\hat{x} \sin[\omega_0 t + \phi_i + \eta_i(t)] + \hat{y} \cos[\omega_0 t + \phi_i + \eta_i(t)] \} \\ &= -\hat{r}_i (\omega_0 + d\eta_i/dt)^2 R + \hat{\phi}_i (d^2\eta_i/dt^2) R \end{aligned} \quad (33)$$

where \hat{r}_i and $\hat{\phi}_i$ are given in Eqs. (20) and (23).

Since the magnet charges e_1 and e_2 are constrained to move in a circular orbit, the perturbation of the charges is determined by the tangential acceleration. For charge e_i , the equation of motion requires only the electrostatic forces due to the stationary charge q and the moving charge $e_{j \neq i}$. There are relativistic fields of order $1/c^2$ in the tangential direction due to the other magnet charges, but these $1/c^2$ -corrections will not contribute to average momentum of the magnet. The radial forces are balanced by the forces of constraint. Thus from Eq. (33), we have for the nonrelativistic equation of motion

$$\begin{aligned} m\mathbf{a}_i \cdot \hat{\phi}_i &\approx mR \frac{d^2\eta_i}{dt^2} \\ &= \hat{\phi}_i \cdot [e\mathbf{E}_q(\mathbf{r}_i, t) + e\mathbf{E}_{j \neq i}(\mathbf{r}_i, t)] = \hat{\phi}_i \cdot \left(\frac{-\hat{x}eq}{x_q^2} + \frac{e^2(\mathbf{r}_i - \mathbf{r}_{j \neq i})}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|^3} \right) \\ &= \left(-\hat{\phi}_i \cdot \hat{x} \frac{eq}{x_q^2} - \hat{\phi}_i \cdot \mathbf{r}_{j \neq i} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|^3} \right) \end{aligned} \quad (34)$$

since $\hat{\phi}_i \cdot \mathbf{r}_i = 0$, where the tangential unit vector is given in Eq. (23) while $\mathbf{r}_{j \neq i} = \hat{x}R \cos[\omega_0 t +$

$\phi_i + \pi + \eta_{j \neq i}(t)] + \hat{y}R \sin[\omega_0 t + \phi_i + \pi + \eta_{j \neq i}(t)]$. Then we have

$$\begin{aligned} \hat{\phi}_i \cdot \mathbf{r}_{j \neq i} &= R \{ -\sin(\omega_0 t + \phi_i + \eta_i) \cos[\omega_0 t + \phi_i + \pi + \eta_{j \neq i}(t)] \\ &\quad + \cos(\omega_0 t + \phi_i + \eta_i) \sin[\omega_0 t + \phi_i + \pi + \eta_{j \neq i}(t)] \} \\ &= R \sin(\eta_{j \neq i} - \eta_i + \pi) = -R \sin(\eta_{j \neq i} - \eta_i) \approx R(\eta_i - \eta_{j \neq i}) \end{aligned} \quad (35)$$

where we have used the approximation $\sin \phi \approx \phi$ for small ϕ . The separation $|\mathbf{r}_i - \mathbf{r}_{j \neq i}|$ between the charges is second order in the perturbation η , and so we may write $|\mathbf{r}_i - \mathbf{r}_{j \neq i}| \approx 2R$ in Eq. (34). Then the nonrelativistic equation of motion (34) for the charge e_i through first order in the perturbation produced by eq/x_q^2 becomes

$$mR \frac{d^2 \eta_i}{dt^2} = \left(\frac{eq}{x_q^2} \sin(\omega_0 t + \phi_i) - \frac{e^2}{(2R)^3} R(\eta_i - \eta_{j \neq i}) \right) \quad (36)$$

We notice that since $\phi_i - \phi_{j \neq i} = \pi$, this equation (36) is odd under the interchange of the two particles. Thus for the steady-state situation, we must have

$$\eta_{j \neq i} = -\eta_i \quad (37)$$

Then the perturbation in the phase η_i (in steady state) is given by

$$mR \frac{d^2 \eta_i}{dt^2} = \left(\frac{eq}{x_q^2} \sin(\omega_0 t + \phi_i) - \frac{e^2}{(2R)^2} \eta_i \right) \quad (38)$$

with a steady-state solution

$$\eta_i(t) = \frac{eq}{x_q^2} \frac{\sin(\omega_0 t + \phi_i)}{[-m\omega_0^2 R + e^2/(2R)^2]} \quad (39)$$

If we take the magnitude e of the charges as small (so that we may neglect the terms in e^2 involving interactions between the charges), then equation (39) agrees exactly with the one-particle-magnet result in Eq. (29). Thus we recover the non-interacting particle result in the appropriate small-charge- e limit. On the other hand, if the magnitude e of the charges becomes large, then according to Eq. (39) the electrostatic interaction contribution $e^2/(2R)^2$ can dominate the mechanical contribution $-m\omega_0^2 R$.

D. Internal Momentum in the Magnet

The internal canonical momentum of the magnet in the presence of the external charge q is given by the sum over the canonical momenta \mathbf{p}_i in Eq. (5) where the sum includes only

the two charges of the magnet in our model, each with canonical momentum

$$\mathbf{p}_{ei} = \mathbf{p}_{ei}^{mech} + \mathbf{p}_{ei}^{field} = m \left(1 + \frac{\mathbf{v}_i^2}{2c^2} \right) \mathbf{v}_i + \frac{e_i e_{j \neq i}}{2c^2} \left(\frac{\mathbf{v}_{j \neq i}}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|} + \frac{\mathbf{v}_{j \neq i} \cdot (\mathbf{r}_i - \mathbf{r}_{j \neq i})(\mathbf{r}_i - \mathbf{r}_{j \neq i})}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|^3} \right) \quad (40)$$

When averaged in time, we expect equal momentum contributions from each charge. The velocity \mathbf{v}_i is given in Eq. (22) where η_i is given in Eq. (39) and its time derivative is

$$\frac{d\eta_i}{dt} = \omega_0 \frac{eq}{x_q^2} \frac{\cos(\omega_0 t + \phi_i)}{[-m\omega_0^2 R + e^2/(2R)^2]} \quad (41)$$

1. Mechanical Linear Momentum of a Perturbed Magnet Charge

Then the mechanical contribution \mathbf{p}_i^{mech} to the linear momentum is

$$\begin{aligned} \mathbf{p}_{ei}^{mech} &= m \left(1 + \frac{v_i^2}{2c^2} \right) \mathbf{v}_i = m\mathbf{v}_i + m \frac{(\mathbf{v}_{0i} + \delta\mathbf{v}_i)^2}{2c^2} \mathbf{v}_i \approx m\mathbf{v}_i + m \frac{\mathbf{v}_{0i}^2}{2c^2} \mathbf{v}_i + m \frac{\mathbf{v}_{0i} \cdot \delta\mathbf{v}_i}{c^2} \mathbf{v}_{0i} \\ &= m\mathbf{v}_i + m \frac{\mathbf{v}_{0i}^2}{2c^2} \mathbf{v}_i + \frac{m}{c^2} \omega_0 \frac{d\eta}{dt} R^2 \mathbf{v}_{0i} \end{aligned} \quad (42)$$

from Eqs. (22) and (24). The average value of the velocity is zero, $\langle \mathbf{v}_i \rangle = 0$, since the magnet-charge interaction is assumed stationary. The required average for \mathbf{p}_i^{mech} follows from Eqs. (12), (41), and (42) as

$$\langle \mathbf{p}_{ei}^{mech} \rangle = \left\langle \frac{m}{c^2} \omega_0 \frac{d\eta}{dt} R^2 \mathbf{v}_{0i} \right\rangle = \hat{y} \frac{m}{c^2} \omega_0^2 R^2 \frac{eq}{x_q^2} \frac{\omega_0 R/2}{[-m\omega_0^2 R + e^2/(2R)^2]} \quad (43)$$

2. Electromagnetic Linear Momentum Associated with a Perturbed Magnet Charge

The electromagnetic contribution \mathbf{p}_i^{field} corresponds to

$$\begin{aligned} \mathbf{p}_{ei}^{field} &= \frac{e_i e_{j \neq i}}{2c^2} \left(\frac{\mathbf{v}_{j \neq i}}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|} + \frac{\mathbf{v}_{j \neq i} \cdot (\mathbf{r}_i - \mathbf{r}_{j \neq i})(\mathbf{r}_i - \mathbf{r}_{j \neq i})}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|^3} \right) \\ &= \frac{e^2}{2c^2} \left(\frac{\mathbf{v}_{j \neq i}}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|} + \frac{\mathbf{v}_{j \neq i} \cdot \mathbf{r}_i (\mathbf{r}_i - \mathbf{r}_{j \neq i})}{|\mathbf{r}_i - \mathbf{r}_{j \neq i}|^3} \right) \end{aligned} \quad (44)$$

since $\mathbf{v}_{j \neq i} \cdot \mathbf{r}_{j \neq i} = 0$. The denominator will involve a distance $2R$ through first order in the perturbation. We need first

$$\mathbf{v}_{j \neq i} \cdot \mathbf{r}_i = v_{ji} \hat{\phi}_{j \neq i} \cdot \mathbf{r}_i \approx \omega_0 R^2 (-2\eta_i) \quad (45)$$

from Eqs. (35) and (37). Then from Eqs. (13), (39), and (45), the time-average of $(\mathbf{v}_{j \neq i} \cdot \mathbf{r}_i) \mathbf{r}_i$ becomes

$$\langle (\mathbf{v}_{j \neq i} \cdot \mathbf{r}_i) \mathbf{r}_i \rangle = \langle \omega_0 R^2 (-2\eta_i) \mathbf{r}_{i0} \rangle = (-2\omega_0 R^2) \frac{eq}{x_q^2} \frac{1}{[-m\omega_0^2 R + e^2/(2R)^2]} \hat{y} \frac{R}{2} \quad (46)$$

and similarly

$$\langle (\mathbf{v}_{j \neq i} \cdot \mathbf{r}_i) \mathbf{r}_{j \neq i} \rangle = (2\omega_0 R^2) \frac{eq}{x_q^2} \frac{1}{[-m\omega_0^2 R + e^2/(2R)^2]} \hat{y} \frac{R}{2} \quad (47)$$

Then from Eq. (44) the time-average electromagnetic contribution to \mathbf{p}_{ei} is

$$\begin{aligned} \langle \mathbf{p}_{ei}^{field} \rangle &= \frac{e^2}{2c^2} \left[(-2\omega_0 R^2) \frac{eq}{x_q^2} \frac{2}{[-m\omega_0^2 R + e^2/(2R)^2]} \hat{y} \frac{R}{2} \left(\frac{1}{(2R)^3} \right) \right] \\ &= -\hat{y} \left(\frac{e^2}{(2R)^2} \right) \frac{\omega_0 R^2}{2c^2} \frac{eq/x_q^2}{[-m\omega_0^2 R + e^2/(2R)^2]} \end{aligned} \quad (48)$$

Adding the mechanical contribution in Eq. (43) and the electromagnetic contribution in Eq. (48), we find

$$\begin{aligned} \langle \mathbf{p}_{ei} \rangle &= \langle \mathbf{p}_{ei}^{mech} \rangle + \langle \mathbf{p}_{ei}^{field} \rangle = \hat{y} \frac{m}{c^2} \omega_0^2 R^2 \frac{eq}{x_q^2} \frac{\omega_0 R/2}{[-m\omega_0^2 R + e^2/(2R)^2]} \\ &\quad - \hat{y} \left(\frac{e^2}{(2R)^2} \right) \frac{\omega_0 R^2}{2c^2} \frac{eq/x_q^2}{[-m\omega_0^2 R + e^2/(2R)^2]} \\ &= -\frac{\hat{y} q e \omega_0 R^2}{2c^2 x_q^2} \end{aligned} \quad (49)$$

The two equal contributions $\langle \mathbf{p}_{ei} \rangle$ and $\langle \mathbf{p}_{ej \neq i} \rangle$ from the two particles give the canonical momentum $\langle \mathbf{p}_{magnet} \rangle$ of the magnet as

$$\langle \mathbf{p}_{magnet} \rangle = \langle \mathbf{p}_{ei} \rangle + \langle \mathbf{p}_{ej \neq i} \rangle = -2 \frac{\hat{y} q e \omega_0 R^2}{2c^2 x_q^2} \quad (50)$$

But then the canonical momentum $\langle \mathbf{p}_{magnet} \rangle$ of the two-particle magnet is equal in magnitude and opposite in sign compared to the canonical momentum $\langle \mathbf{p}_q \rangle$ of the external charge q corresponding to $N = 2$ in Eq. (18), which was equal to the familiar electromagnetic field momentum involving the electric field due to q and the magnetic field of the magnet. We see that the relativistic conservation law (10) regarding the center of energy is indeed satisfied and the total momentum of the system indeed vanishes.

E. Discussion of Internal Electromagnetic Momentum

Following equation (39), we noted that in the limit of small value for the charge e of the magnet particles, the mechanical momentum dominated the internal momentum of the magnet. This mechanical momentum corresponds to the “hidden mechanical momentum” of the textbooks and literature. However, in the opposite limit of large charge e for the magnet particles, the electromagnetic momentum becomes large and the mechanical momentum

becomes negligible. As more particles of fixed mass m and charge e are added to the magnet while keeping the magnetic moment $\vec{\mu}$ fixed, the speed $v_{0i} = \omega_0 R$ of the current carriers becomes ever smaller so that the mechanical momentum becomes insignificant compared to the internal electromagnetic momentum. Thus for any physical multiparticle magnet with its enormous number of charge carriers, we expect that only the internal electromagnetic momentum needs to be considered. This internal electromagnetic momentum is equal in magnitude and opposite in direction from the electromagnetic field momentum which is found in the elementary textbook calculations involving a point charge and a steady current. The negligible contribution of the particle mechanical momentum is analogous to the negligible contribution of the particle kinetic energy to the self-inductance of a circuit[16] where the mass and the charge of the charge carriers is never mentioned in the textbooks.

Although Coleman and Van Vleck[3] insisted that the “hidden momentum” of Shockley and James[2] was purely mechanical, other authors (such as Furry[17]) have been more cautious. In the literature, “hidden mechanical momentum” is invoked simply as what is left over after the electromagnetic momentum has been calculated. However, there seems to be no recognition of the separation between the external and internal electromagnetic momentum for a magnet interacting with a point charge. The external electromagnetic momentum is familiar from all the textbooks as the momentum associated with the electric field of the external charge and the magnetic field of the unperturbed magnet. The unrecognized internal electromagnetic momentum involves the field momentum between the electric and magnetic fields of the magnet particles undergoing perturbed motion due to the electric field of the external point charge.

To some readers the shift presented here from a “hidden mechanical momentum” over to an “internal electromagnetic momentum” may seem like a distinction without a difference. In both cases, we are describing linear momentum which is internal to the magnet. The crucial difference arises in the implications of the two different forms of momentum. Thus “hidden mechanical momentum” has been used as an excuse to claim that a charge passing a magnet experienced no forces[18] (which forces might lead to an electromagnetic lag effect), and has been used to claim that a magnetic moment passing an electric line charge (which magnetic moment experienced an obvious Lorentz force) nevertheless moved as though it experienced no force at all.[19][20] These claims have been accepted into the mainstream physics literature both in research and in the textbooks. There seems to be no recognition

that changes in the external and internal electromagnetic momenta are associated with different types of forces. Changes in the external electromagnetic momentum are associated with *magnetic* forces of order $1/c^2$. Changes in the internal electromagnetic momentum are associated with *electrical* forces of order $1/c^2$. It is these electrical forces associated with changes in the internal electromagnetic momentum which play a crucial role in the experimentally observed Aharonov-Bohm and Aharonov-Casher phase shifts. A discussion of the classical electromagnetic forces responsible for the Aharonov-Bohm phase shift and the Aharonov-Casher phase shift will be given in another publication.[21] Here we note simply that our clarification involving the myth of “hidden mechanical momentum” alters our fundamental understanding of one of the connections between classical and quantum theories.[22]

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- [22] The present article is the second in a series of four articles by T. H. Boyer using the magnet model consisting of mutually-interacting point charges moving on a circular path. Article 1: “Self-Inductance and the Mass of Current Carriers in a Circuit.” Article 3: “Classical Interaction of a Magnet and a Point Charge: The Shockley-James Paradox.” Article 4: “Classical Interaction of a Magnet and a Point Charge: The Classical Electromagnetic Forces Responsible for the Aharonov-Bohm Phase Shift.”